Lec 8

8·1 calculating function of a square matrix.

Ex: $(\lambda-1)(\lambda-2) = \lambda^2 - 3\lambda + 2$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

Q: What is A^{98} ??

$$A^{98} = \alpha_0 I + \alpha_1 A$$

Substituting the eigenvalues we get

$1^{98} = \alpha_0 + \alpha_1 1$ $\qquad \alpha_1 = 2^{98} - 1^{98}$

$2^{98} = \alpha_0 + 2\alpha_1$ $\qquad \boxed{\alpha_1 = 2^{98} - 1}$

$$\alpha_0 = 1 - \alpha_1 = 2 - 2^{98}$$

$$A^{98} = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1 \\ -2\alpha_1 & 3\alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 & \alpha_1 \\ -2\alpha_1 & \alpha_0 + 3\alpha_1 \end{pmatrix}$$

8.2

$$\begin{pmatrix} 2-2^{98} & 2^{98}-1 \\ 2-2^{99} & 2-2^{98}+3\cdot 2^{98}-3 \end{pmatrix}$$

$$= -1+2\cdot 2^{98}$$

$$= 2^{99}-1$$

$$\begin{pmatrix} 2(1-2^{97}) & 2^{98}-1 \\ 2(1-2^{98}) & 2^{99}-1 \end{pmatrix}$$

$$= A^{98}$$

Ex: Repeated eigenvalue:

$\lambda = \lambda_0, \lambda_0$

$(\lambda - \lambda_0)^2 = \lambda^2 - 2\lambda_0\lambda + \lambda_0^2$

$$A = \begin{pmatrix} 0 & 1 \\ -\lambda_0^2 & 2\lambda_0 \end{pmatrix}$$

what is A^{98}??

$A^{98} = \alpha_0 I + \alpha_1 A$

Substituting λ_0 for A we get

$\lambda_0^{98} = \alpha_0 + \alpha_1 \lambda_0$

& $\left.\frac{d}{d\lambda}\lambda^{98}\right|_{\lambda=\lambda_0} = \left.\frac{d}{d\lambda}\left(\alpha_0 + \alpha_1 \lambda\right)\right|_{\lambda=\lambda_0}$

$\Rightarrow 98\lambda_0^{97} = \alpha_1$

$\alpha_0 = \lambda_0^{98} - 98\lambda_0^{97} \cdot \lambda_0 = \lambda_0^{98}(1-98)$

$= -97\lambda_0^{98}$

(8.4)

$$\alpha_0 = -97\lambda_0^{98}$$
$$\alpha_1 = 98\lambda_0^{97}$$

$$A^{98} = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1 \\ -\lambda_0^2\alpha_1 & 2\lambda_0\alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 & \alpha_1 \\ -\lambda_0^2\alpha_1 & \alpha_0 + 2\lambda_0\alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} -97\lambda_0^{98} & 98\lambda_0^{97} \\ \underset{\displaystyle -98\lambda_0^{99}}{\underset{\shortparallel}{-\lambda_0^2(98\lambda_0^{97})}} & \underset{\displaystyle 99\lambda_0^{98}}{\underset{\shortparallel}{-97\lambda_0^{98} + 2\lambda_0(98\lambda_0^{97})}} \end{pmatrix}$$

$$\begin{array}{r} 196 \\ -97 \\ \hline 99 \end{array}$$

8.5

$$\begin{pmatrix} -97\lambda_0^{98} & 98\lambda_0^{97} \\ -98\lambda_0^{99} & 99\lambda_0^{98} \end{pmatrix}$$

$$= \lambda_0^{97} \begin{pmatrix} -97\lambda_0 & 98 \\ -98\lambda_0^{2} & 99\lambda_0 \end{pmatrix}$$

Ex: A 4X4 matrix A has eigenvalues at 3, 3, 4, 4. Calculate e^{At}.

Ans: $e^{At} =$

$$\alpha_0 \underline{I} + \alpha_1 A + \alpha_2 A^2 + \alpha_3 A^3$$

Replacing A by λ we get

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \alpha_3 \lambda^3$$

Differentiating w.r.t. λ we get

$$t e^{\lambda t} = 0 + \alpha_1 + 2\alpha_2 \lambda + 3\alpha_3 \lambda^2$$

For $\lambda = 3$ we get

$$e^{3t} = \alpha_0 + 3\alpha_1 + 9\alpha_2 + 27\alpha_3$$

$$te^{3t} = \alpha_1 + 6\alpha_2 + 27\alpha_3$$

For $\lambda = 4$ we get

$$e^{4t} = \alpha_0 + 4\alpha_1 + 16\alpha_2 + 64\alpha_3$$

$$te^{4t} = \alpha_1 + 8\alpha_2 + 48\alpha_3$$

generalized Vander Monde Matrix

$$\begin{pmatrix} 1 & 3 & 9 & 27 \\ 0 & 1 & 6 & 27 \\ 1 & 4 & 16 & 64 \\ 0 & 1 & 8 & 48 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ te^{3t} \\ e^{4t} \\ te^{4t} \end{pmatrix}$$

$$\alpha_0 = -80e^{3t} - 48te^{3t} + 81e^{4t} - 36te^{4t}$$

$$\alpha_1 = 72e^{3t} + 40te^{3t} - 72e^{4t} + 33te^{4t}$$

$$\alpha_2 = -21e^{3t} - 11te^{3t} + 21e^{4t} - 10te^{4t}$$

$$\alpha_3 = 2e^{3t} + te^{3t} - 2e^{4t} + te^{4t}.$$

Various Van Der Monde Matrices of size 4×4

8.8

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{pmatrix} \leftarrow \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4.$$

If $\lambda_1 = \lambda_2$ $\lambda_2 \neq \lambda_3 \neq \lambda_4$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ \lambda_1 & 1 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_3^3 & \lambda_4^3 \end{pmatrix} \leftarrow \begin{matrix} \lambda_1 = \lambda_2 \\ \lambda_2 \neq \lambda_3 \neq \lambda_4 \end{matrix}$$

If $\lambda_1 = \lambda_2 = \lambda_3$ $\lambda_3 \neq \lambda_4$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ \lambda_1 & 1 & 0 & \lambda_4 \\ \lambda_1^2 & 2\lambda_1 & 2 & \lambda_4^2 \\ \lambda_1^3 & 3\lambda_1^2 & 6\lambda_1 & \lambda_4^3 \end{pmatrix}$$

Q: A 2×2 matrix A has eigenvalues at $(\cos\theta \pm i\sin\theta)\cdot r$ where r, θ are fixed. Calculate A^N.

Ans: Writing

$$A^N = \alpha_0 I + \alpha_1 A \quad , N \geq 2$$

we have

$$[r(\cos\theta + i\sin\theta)]^N =$$

$$\alpha_0 + \alpha_1 r[\cos\theta + i\sin\theta]$$

$$\Updownarrow$$

Applying De Moivre's Theorem.

$$r^N(\cos N\theta + i\sin N\theta) =$$

$$(\alpha_0 + \alpha_1 r\cos\theta) + i(\alpha_1 r\sin\theta)$$

Solving for α_0 and α_1 we obtain

$$\alpha_1 = \frac{\sin N\theta}{\sin\theta} r^{N-1}$$

$$\alpha_0 = -\frac{r^N}{\sin\theta} \sin[(N-1)\theta]$$

— x —

Q: Calculate the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

This is a 3x3 skew symmetric matrix.

Ans: $0, i\omega, -i\omega$ where

$$\omega = \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9}$$
$$= \sqrt{14}$$

Q: Assume that A is 3×3 and has eigenvalues at $0, i\omega, -i\omega$. Calculate e^{At}.

A: $$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2.$$

Eigenvalue 0 gives

$$1 = \alpha_0$$

Eigenvalue $i\omega$ gives

$$e^{i\omega t} = \alpha_0 + \alpha_1(i\omega) + \alpha_2(i\omega)^2$$

$$e^{i\omega t} = (\alpha_0 - \alpha_2\omega^2) + i(\alpha_1\omega) \quad (1)$$

Eigenvalue $-i\omega$ gives

$$e^{-i\omega t} = (\alpha_0 - \alpha_2\omega^2) - i(\alpha_1\omega) \quad (2)$$

Adding (1) and (2) we get

$$\underbrace{e^{i\omega t} + e^{-i\omega t}}_{2\cos\omega t} = 2(\alpha_0 - \alpha_2\omega^2)$$

$$\alpha_0 - \alpha_2 \omega^2 = \cos\omega t$$

$$\alpha_2 \omega^2 = 1 - \cos\omega t.$$

$$\boxed{\alpha_2 = \frac{1}{\omega^2} - \frac{\cos\omega t}{\omega^2}}$$

Subtracting (2) from (1) we get.

$$e^{i\omega t} - e^{-i\omega t} = 2i\alpha_1\omega$$

$$\cancel{i}\,\cancel{2}\sin\omega t = \cancel{i}\,\cancel{2}\alpha_1\omega$$

$$\alpha_1 = \frac{\sin\omega t}{\omega}.$$

Hence $\alpha_0 = 1$, $\alpha_1 = \frac{\sin\omega t}{\omega}$, $\alpha_2 = \frac{1-\cos\omega t}{\omega^2}$

$$\boxed{e^{At} = I + \frac{\sin\omega t}{\omega}A + \frac{1-\cos\omega t}{\omega^2}A^2}$$

Symmetric Matrices

Def: A matrix A is symmetric if $A = A^T$.

Ex: $\begin{pmatrix} 1 & 3 & 6 & 8 \\ 3 & 2 & 7 & 9 \\ 6 & 7 & 5 & -1 \\ 8 & 9 & -1 & 9 \end{pmatrix}$

① Eigenvalues of a symmetric matrix are always real.

② A symmetric $n \times n$ matrix always has n l.i. eigenvectors.

Remark: No need to talk about generalized eigenvectors.

Example:

```
>> A=[1 3 6 8;3 2 7 9;6 7 5 -1;8 9 -1 9]

A =
```

$$\begin{pmatrix} 1 & 3 & 6 & 8 \\ 3 & 2 & 7 & 9 \\ 6 & 7 & 5 & -1 \\ 8 & 9 & -1 & 9 \end{pmatrix}$$

A symmetric 4×4 matrix.

```
>> [v1 v2]=eig(A)

v1 =
```

$$\begin{pmatrix} -0.4834 & 0.7506 & -0.0830 & 0.4427 \\ -0.5394 & -0.6605 & -0.1193 & 0.5085 \\ 0.4848 & 0.0177 & -0.8017 & 0.3492 \\ 0.4902 & -0.0041 & 0.5798 & 0.6508 \end{pmatrix}$$

The columns are eigenvectors.

```
v2 =
```

$$\begin{pmatrix} -9.7820 & 0 & 0 & 0 \\ 0 & -1.5416 & 0 & 0 \\ 0 & 0 & 7.3859 & 0 \\ 0 & 0 & 0 & 20.9376 \end{pmatrix}$$

```
>> inv(v1)*A*v1

ans =
```

$$\begin{pmatrix} -9.7820 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -1.5416 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 7.3859 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 20.9376 \end{pmatrix}$$

```
>>
```

Two eigenvalues of A are positive
and two " " " " negative.

Def: A symmetric matrix A is positive definite, if all its real eigenvalues are positive; positive semidefinite if all its real eigenvalues are non-negative;

Like wise one can define negative definite and negative semidefinite.

It is indefinite if there is atleast one positive and one negative eigenvalue.

The matrix in the example on page 8.14 is indefinite.

Fact 1

Eigenvalues of a symmetric matrix A must be real.

Proof: Let λ be an eigenvalue of A where x is an eigenvector, we have

$$Ax = \lambda x \qquad (*)$$

Taking complex conjugate of $(*)$, we obtain

$$A\bar{x} = \bar{\lambda}\bar{x} \qquad (**)$$

From $(*)$ we write

$$\bar{x}^T A x = \bar{x}^T \lambda x = \lambda \bar{x}^T x$$

From $(**)$ we write

$$x^T A \bar{x} = x^T \bar{\lambda} \bar{x} = \bar{\lambda} x^T \bar{x}$$

Note that

$$\bar{x}^T A x = x^T A^T \bar{x} \quad \text{(because it is a scalar)}$$
$$= x^T A \bar{x} \quad (\text{because } A = A^T).$$

It follows that

$$\lambda \bar{x}^T x = \bar{\lambda} x^T \bar{x}$$

But

$$\bar{x}^T x = x^T \bar{x}$$

hence

$$\lambda = \bar{\lambda}.$$

Thus every eigenvalue of A is equal to its conjugate. Hence λ is real.

Fact 2

Eigenvectors of A corresponding to distinct eigenvalues are mutually orthogonal if $A = A^T$.

proof

$$\begin{matrix} \lambda_1 & v_1 \\ \lambda_2 & v_2 \end{matrix}$$

← be two eigenvalue eigenvector pairs.

(8.18)

$$A v_1 = \lambda_1 v_1$$
$$A v_2 = \lambda_2 v_2 .$$

It follows that

$$v_1^T A v_2 = v_1^T \lambda_2 v_2 = \lambda_2 v_1^T v_2 .$$
$$v_2^T A v_1 = v_2^T \lambda_1 v_1 = \lambda_1 v_2^T v_1$$

Once again, since $v_1^T A v_2 = v_2^T A v_1$
(Follows from $A = A^T$)

we have

$$\lambda_2 v_1^T v_2 = \lambda_1 v_2^T v_1 = \lambda_1 v_1^T v_2 .$$

$$\Rightarrow (\lambda_2 - \lambda_1) v_1^T v_2 = 0$$

$\because \lambda_1 \neq \lambda_2$ we have $v_1^T v_2 = 0$

Thus $v_1 \cdot v_2 = 0$ and v_1 & v_2 are perpendicular.

$\blacksquare$

Orthogonal Matrices :-

A $n \times n$ real matrix Q is said to be orthogonal matrix if

$$Q^T Q = I$$

Ex: 2×2 orthogonal matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Ex: Manufacture a 3x3 orthogonal matrix.

A =

$$\begin{pmatrix} 2 & 4 & 7 \\ 8 & 9 & -5 \\ 4 & -6 & -12 \end{pmatrix}$$

```
>> B=A-transpose(A)
```

B =

$$\begin{pmatrix} 0 & -4 & 3 \\ 4 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix}$$

Skew symmetric matrix.

$B = A - A^T$

Show that $B^T = -B$

```
>> C=expm(B)
```

C =

$c = e^B$

$$\begin{pmatrix} 0.4010 & 0.6547 & -0.6408 \\ -0.7984 & 0.5927 & 0.1059 \\ 0.4491 & 0.4691 & 0.7604 \end{pmatrix}$$

orthogonal matrix

Show that $CC^T = I$

```
>> C*transpose(C)
```

ans =

$$\begin{pmatrix} 1.0000 & 0.0000 & 0 \\ 0.0000 & 1.0000 & 0.0000 \\ 0 & 0.0000 & 1.0000 \end{pmatrix}$$

```
>> det(C)
```

ans =

1.0000

```
>> eig(C)
```

ans =

0.3771 + 0.9262i
0.3771 - 0.9262i
1.0000

The eigenvalues have unit modulus.

Ex: Manufacture a 4x4 orthogonal matrix

```
>> B=A-transpose(A)
```

A =

$$\begin{pmatrix} 2 & 4 & 7 & -4 \\ 8 & 9 & -5 & -3 \\ 4 & -6 & -12 & -20 \\ 3 & 6 & 17 & 29 \end{pmatrix}$$

B =

$$\begin{pmatrix} 0 & -4 & 3 & -7 \\ 4 & 0 & 1 & -9 \\ -3 & -1 & 0 & -37 \\ 7 & 9 & 37 & 0 \end{pmatrix}$$

$B^T = -B$ $\qquad$ $B = A - A^T$

skew symmetric matrix.

```
>> C=expm(B)
```

$C = e^B$

C =

$$\begin{pmatrix} -0.3440 & 0.9236 & -0.0184 & -0.1682 \\ -0.8511 & -0.3308 & 0.3901 & -0.1186 \\ 0.3075 & -0.0373 & 0.3734 & -0.8744 \\ 0.2506 & 0.1901 & 0.8414 & 0.4393 \end{pmatrix}$$

```
>> C*transpose(C)
```

ans =

$$\begin{pmatrix} 1.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 1.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 1.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0000 & 1.0000 \end{pmatrix} = \boxed{CC^T = I}$$

```
>> eig(C)
```

ans =

```
  -0.3743 + 0.9273i
  -0.3743 - 0.9273i
   0.4433 + 0.8964i
   0.4433 - 0.8964i
```

Eigenvalues have unit modulus.

```
>> det(C)
```

ans =

```
    1.0000
```

Ex: Manufacture a 4X4 positive definite matrix :-

```
>> L=[1 0 0 0;0 3 0 0;0 0 5 0;0 0 0 16]

L =
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}$$

```
>> PD=C*L*transpose(C)
```

C is from page 8·2[illegible]

```
PD =
```

$$\begin{pmatrix} 3.1317 & -0.3405 & 2.1095 & -0.8191 \\ -0.3405 & 2.0388 & 2.1633 & 0.4056 \\ 2.1095 & 2.1633 & 13.0298 & -4.5200 \\ -0.8191 & 0.4056 & -4.5200 & 6.7997 \end{pmatrix}$$

CLC^T is a symmetric and positive definite.

— x —

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Define $\Delta_j = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ - & - & - & - \\ a_{j1} & a_{j2} & \cdots & a_{jj} \end{pmatrix}$

Thm:

A is positive definite iff

$\Delta_j > 0 \quad j = 1, 2, \ldots, n.$

Ex: We want to verify the above theorem for the positive definite matrix PD on page 8·22.

Delta1 =

3.1317

Delta2 =

$$\begin{vmatrix} 3.1317 & -0.3405 \\ -0.3405 & 2.0388 \end{vmatrix} = 6.2689$$

Delta3 =

$$\begin{vmatrix} 3.1317 & -0.3405 & 2.1095 \\ -0.3405 & 2.0388 & 2.1633 \\ 2.1095 & 2.1633 & 13.0298 \end{vmatrix} = 54.8465$$

Delta4 =

$$\begin{vmatrix} 3.1317 & -0.3405 & 2.1095 & -0.8191 \\ -0.3405 & 2.0388 & 2.1633 & 0.4056 \\ 2.1095 & 2.1633 & 13.0298 & -4.5200 \\ -0.8191 & 0.4056 & -4.5200 & 6.7997 \end{vmatrix} = 240$$

Properties of orthogonal Matrices :-

Theorem :

If Q is an orthogonal matrix then $\det Q = \pm 1$

Proof:

$$Q \cdot Q^T = I$$

$$\det(Q\,Q^T) = \det I = 1$$

$$\det Q \cdot \det Q^T = 1$$

$$\det Q \cdot \det Q = 1$$

$$(\det Q)^2 = 1$$

$$\boxed{\det Q = \pm 1}$$

Theorem: Product of two orthogonal matrices is an orthogonal matrix.

Theorem:

Eigenvalues of an orthogonal matrix are all of unit modulus.

proof: Let λ be an eigenvalue and v be the corresponding eigenvector.

$$Qv = \lambda v$$
$$Q\bar{v} = \bar{\lambda}\bar{v}$$

$$\bar{v}^T Q^T Q v = \bar{\lambda}\, \bar{v}^T v\, \lambda$$
$$\bar{v}^T v = \bar{\lambda}\lambda\, \bar{v}^T v$$
$$\bar{\lambda}\lambda = 1 \Rightarrow |\lambda|^2 = 1$$

Example:

A =

$$\begin{pmatrix} 9 & 4 & 0 & 5 & 4 & 7 & 1 & 2 & 4 & 4 & 4 & 0 & 1 & 1 & 9 \\ 2 & 9 & 7 & 7 & 8 & 5 & 0 & 5 & 3 & 7 & 0 & 3 & 4 & 8 & 2 \\ 6 & 9 & 4 & 4 & 8 & 4 & 8 & 7 & 8 & 2 & 0 & 6 & 7 & 1 & 1 \\ 4 & 4 & 9 & 3 & 6 & 6 & 1 & 5 & 0 & 4 & 3 & 7 & 8 & 1 & 8 \\ 8 & 8 & 4 & 1 & 8 & 6 & 2 & 6 & 7 & 9 & 0 & 6 & 2 & 9 & 2 \\ 7 & 0 & 4 & 1 & 6 & 7 & 6 & 2 & 9 & 6 & 3 & 0 & 2 & 4 & 6 \\ 4 & 3 & 8 & 6 & 3 & 9 & 2 & 3 & 9 & 2 & 6 & 4 & 8 & 3 & 9 \\ 0 & 8 & 5 & 3 & 2 & 5 & 4 & 7 & 7 & 8 & 0 & 4 & 2 & 3 & 6 \\ 8 & 0 & 2 & 5 & 3 & 8 & 0 & 6 & 4 & 6 & 0 & 3 & 8 & 3 & 8 \\ 4 & 1 & 6 & 1 & 5 & 1 & 9 & 4 & 4 & 1 & 6 & 1 & 9 & 3 & 0 \\ 6 & 2 & 8 & 6 & 7 & 9 & 5 & 5 & 2 & 2 & 6 & 6 & 2 & 5 & 1 \\ 7 & 1 & 0 & 3 & 3 & 2 & 4 & 7 & 6 & 6 & 0 & 6 & 2 & 1 & 8 \\ 9 & 6 & 6 & 8 & 8 & 2 & 5 & 0 & 3 & 6 & 0 & 7 & 0 & 0 & 4 \\ 7 & 2 & 3 & 8 & 5 & 8 & 3 & 6 & 9 & 3 & 1 & 4 & 0 & 4 & 8 \\ 1 & 1 & 8 & 5 & 3 & 7 & 4 & 0 & 7 & 5 & 5 & 5 & 6 & 8 & 7 \end{pmatrix}$$

Sym=A+transpose(A)

Sym =

$$\begin{pmatrix} 18 & 6 & 6 & 9 & 12 & 14 & 5 & 2 & 12 & 8 & 10 & 7 & 10 & 8 & 10 \\ 6 & 18 & 16 & 11 & 16 & 5 & 3 & 13 & 3 & 8 & 2 & 4 & 10 & 10 & 3 \\ 6 & 16 & 8 & 13 & 12 & 8 & 16 & 12 & 10 & 8 & 8 & 6 & 13 & 4 & 9 \\ 9 & 11 & 13 & 6 & 7 & 7 & 7 & 8 & 5 & 5 & 9 & 10 & 16 & 9 & 13 \\ 12 & 16 & 12 & 7 & 16 & 12 & 5 & 8 & 10 & 14 & 7 & 9 & 10 & 14 & 5 \\ 14 & 5 & 8 & 7 & 12 & 14 & 15 & 7 & 17 & 7 & 12 & 2 & 4 & 12 & 13 \\ 5 & 3 & 16 & 7 & 5 & 15 & 4 & 7 & 9 & 11 & 11 & 8 & 13 & 6 & 13 \\ 2 & 13 & 12 & 8 & 8 & 7 & 7 & 14 & 13 & 12 & 5 & 11 & 2 & 9 & 6 \\ 12 & 3 & 10 & 5 & 10 & 17 & 9 & 13 & 8 & 10 & 2 & 9 & 11 & 12 & 15 \\ 8 & 8 & 8 & 5 & 14 & 7 & 11 & 12 & 10 & 2 & 8 & 7 & 15 & 6 & 5 \\ 10 & 2 & 8 & 9 & 7 & 12 & 11 & 5 & 2 & 8 & 12 & 6 & 2 & 6 & 6 \\ 7 & 4 & 6 & 10 & 9 & 2 & 8 & 11 & 9 & 7 & 6 & 12 & 9 & 5 & 13 \\ 10 & 10 & 13 & 16 & 10 & 4 & 13 & 2 & 11 & 15 & 2 & 9 & 0 & 0 & 10 \\ 8 & 10 & 4 & 9 & 14 & 12 & 6 & 9 & 12 & 6 & 6 & 5 & 0 & 8 & 16 \\ 10 & 3 & 9 & 13 & 5 & 13 & 13 & 6 & 15 & 5 & 6 & 13 & 10 & 16 & 14 \end{pmatrix}$$

15 × 15 symmetric matrix.

Matrix is not positive definite.

15 real eigenvalues.

eig(Sym)

ans =

-23.9196, -18.4932, -12.2937, -11.1462, -7.3854, -6.6783, -3.7384, 2.9757, 9.1224, 9.6423, 13.4695, 16.9037, 19.9091, 29.9777, 135.6544

set of 15 orthogonal eigenvectors.

$$P = \begin{pmatrix} -0.0338 & -0.0262 & 0.0128 & 0.2404 & 0.1701 & -0.2284 & 0.0050 & 0.5360 & -0.0642 & -0.1704 & -0.3946 & -0.3450 & 0.3554 & -0.2614 & 0.2633 \\ -0.1196 & -0.1025 & 0.0313 & -0.2688 & -0.3558 & -0.3613 & 0.1545 & 0.1663 & 0.0050 & 0.3488 & -0.0581 & -0.0005 & 0.2115 & 0.6028 & 0.2453 \\ 0.0877 & 0.4670 & 0.0739 & 0.0164 & 0.5923 & 0.0137 & 0.2889 & 0.0624 & 0.2236 & 0.1624 & 0.2274 & -0.1788 & -0.1792 & 0.2406 & 0.2829 \\ -0.3723 & -0.0073 & 0.0530 & 0.1475 & -0.1209 & 0.6374 & -0.2295 & 0.1354 & -0.1729 & 0.3545 & -0.1596 & -0.1970 & -0.2320 & 0.0785 & 0.2549 \\ -0.0776 & -0.2437 & -0.2864 & 0.0984 & 0.2334 & 0.2455 & 0.1619 & -0.5412 & -0.0606 & -0.2250 & -0.1428 & -0.0214 & 0.4332 & 0.2377 & 0.2993 \\ 0.1684 & 0.2586 & -0.1180 & 0.1076 & -0.4935 & 0.2508 & 0.3317 & 0.0456 & 0.2140 & 0.0893 & 0.3118 & 0.0679 & 0.3087 & -0.3560 & 0.2881 \\ -0.2816 & -0.4619 & 0.2453 & 0.4087 & -0.0502 & -0.3149 & 0.0788 & -0.1900 & 0.2158 & 0.0164 & 0.3403 & -0.1523 & -0.2557 & -0.1433 & 0.2539 \\ 0.1925 & -0.2001 & -0.2530 & 0.2466 & 0.0689 & 0.0893 & -0.1441 & 0.4434 & -0.1610 & -0.2669 & 0.3054 & 0.4764 & -0.1350 & 0.2694 & 0.2446 \\ -0.2541 & -0.1861 & 0.2859 & -0.5901 & 0.1412 & 0.1762 & 0.0017 & 0.1374 & 0.3579 & -0.2342 & -0.0723 & 0.3053 & 0.0018 & -0.1993 & 0.2826 \\ -0.2724 & 0.5032 & -0.0806 & 0.0051 & -0.2175 & -0.2073 & -0.5319 & -0.1745 & 0.1047 & -0.4050 & 0.0900 & -0.0914 & -0.0045 & 0.1157 & 0.2399 \\ 0.1040 & -0.1184 & -0.0054 & -0.4382 & 0.0841 & -0.0410 & -0.1149 & -0.0259 & -0.5771 & 0.0122 & 0.4731 & -0.3464 & 0.0702 & -0.1961 & 0.2002 \\ 0.0264 & 0.1779 & 0.2164 & 0.0051 & -0.2178 & -0.0591 & 0.4801 & -0.1159 & -0.4386 & -0.3395 & -0.3004 & 0.0975 & -0.4107 & -0.0042 & 0.2235 \\ 0.6890 & -0.2073 & 0.0425 & -0.0969 & -0.1652 & 0.0951 & -0.2270 & -0.0902 & 0.2993 & -0.0735 & -0.2111 & -0.3279 & -0.2423 & 0.1046 & 0.2417 \\ 0.2483 & 0.1198 & 0.5239 & 0.1871 & 0.1234 & -0.0984 & -0.3074 & -0.2186 & -0.1999 & 0.3108 & -0.0943 & 0.4107 & 0.2539 & -0.0964 & 0.2438 \\ -0.0318 & 0.0108 & -0.5992 & -0.1186 & 0.0968 & -0.2873 & -0.0642 & -0.1255 & 0.0043 & 0.3619 & -0.2371 & 0.2186 & -0.2822 & -0.3407 & 0.2885 \end{pmatrix}$$

orthogonal matrix of eigenvectors.

The matrix P is an orthogonal matrix.

$\det P = -1$

$Q = -P$ is also orthogonal

$\det Q = (-1)^{15} \det P = 1.$

Eigenvalues of Q have unit magnitude and are located at

Eigenvalues have unit magnitude:

1.0000
0.9710 + 0.2389i
0.9710 - 0.2389i
0.5875 + 0.8093i
0.5875 - 0.8093i
0.0507 + 0.9987i
0.0507 - 0.9987i
-0.9943 + 0.1065i
-0.9943 - 0.1065i
-0.8506 + 0.5258i
-0.8506 - 0.5258i
-0.4628 + 0.8865i
-0.4628 - 0.8865i
-0.5839 + 0.8118i
-0.5839 - 0.8118i

15X15 skew symmetric matrix

All eigenvalues have zero real part

```
>> Skew=A-transpose(A)

Skew =
```

$$\begin{pmatrix}
0 & 2 & -6 & 1 & -4 & 0 & -3 & 2 & -4 & 0 & -2 & -7 & -8 & -6 & 8 \\
-2 & 0 & -2 & 3 & 0 & 5 & -3 & -3 & 3 & 6 & -2 & 2 & -2 & 6 & 1 \\
6 & 2 & 0 & -5 & 4 & 0 & 0 & 2 & 6 & -4 & -8 & 6 & 1 & -2 & -7 \\
-1 & -3 & 5 & 0 & 5 & 5 & -5 & 2 & -5 & 3 & -3 & 4 & 0 & -7 & 3 \\
4 & 0 & -4 & -5 & 0 & 0 & -1 & 4 & 4 & 4 & -7 & 3 & -6 & 4 & -1 \\
0 & -5 & 0 & -5 & 0 & 0 & -3 & -3 & 1 & 5 & -6 & -2 & 0 & -4 & -1 \\
3 & 3 & 0 & 5 & 1 & 3 & 0 & -1 & 9 & -7 & 1 & 0 & 3 & 0 & 5 \\
-2 & 3 & -2 & -2 & -4 & 3 & 1 & 0 & 1 & 4 & -5 & -3 & 2 & -3 & 6 \\
4 & -3 & -6 & 5 & -4 & -1 & -9 & -1 & 0 & 2 & -2 & -3 & 5 & -6 & 1 \\
0 & -6 & 4 & -3 & -4 & -5 & 7 & -4 & -2 & 0 & 4 & -5 & 3 & 0 & -5 \\
2 & 2 & 8 & 3 & 7 & 6 & -1 & 5 & 2 & -4 & 0 & 6 & 2 & 4 & -4 \\
7 & -2 & -6 & -4 & -3 & 2 & 0 & 3 & 3 & 5 & -6 & 0 & -5 & -3 & 3 \\
8 & 2 & -1 & 0 & 6 & 0 & -3 & -2 & -5 & -3 & -2 & 5 & 0 & 0 & -2 \\
6 & -6 & 2 & 7 & -4 & 4 & 0 & 3 & 6 & 0 & -4 & 3 & 0 & 0 & 0 \\
-8 & -1 & 7 & -3 & 1 & 1 & -5 & -6 & -1 & 5 & 4 & -3 & 2 & 0 & 0
\end{pmatrix}$$

```
>> eig(Skew)
```

+27.9097i, -27.9097i
+19.9418i, -19.9418i
+16.9653i, -16.9653i
+12.4824i, -12.4824i
+ 7.2754i, - 7.2754i
+ 4.5154i, - 4.5154i
+ 2.5339i, - 2.5339i
0.0000

-0.1652	-0.4566	-0.0490	0.1815	-0.0286	-0.1673	0.0270	-0.2750	0.3122	0.2372	0.5849	-0.1955	-0.1329	0.0224	0.2674
0.2557	0.1157	-0.2323	-0.0251	0.4657	-0.1047	-0.0729	-0.0828	-0.0123	0.0594	0.0802	-0.2576	0.5730	-0.4442	0.1566
0.1335	-0.0757	-0.0658	-0.3220	0.1203	0.2295	0.1262	0.4794	0.2259	0.3903	-0.0210	-0.0879	-0.4317	-0.3968	-0.0133
0.0758	0.1650	0.2744	0.5077	0.1402	0.2308	-0.3610	0.0727	-0.2780	0.5391	-0.0487	-0.0881	-0.0643	0.1445	0.1492
0.0911	-0.0161	0.1905	-0.0844	-0.2449	-0.6156	-0.2757	-0.2893	0.0307	0.2130	-0.3808	0.0166	-0.1862	-0.3509	-0.0027
-0.0960	-0.3056	0.1925	-0.5254	0.1810	-0.0036	-0.2086	0.0633	-0.4218	-0.1071	-0.0339	-0.0155	-0.0300	0.1577	0.5361
0.4518	0.0794	0.1765	0.2942	-0.1056	-0.0089	0.4505	-0.0032	0.0104	-0.3107	-0.1405	-0.2081	-0.2148	-0.0580	0.4985
-0.2860	-0.1306	-0.0097	0.2078	-0.3196	-0.1370	0.3328	0.3228	-0.3939	0.1607	0.0996	0.3266	0.2714	-0.3625	0.1396
-0.6559	0.5235	-0.1275	0.0635	0.1172	-0.1512	-0.0472	0.1132	0.1077	-0.1430	-0.0427	-0.2509	-0.2070	-0.0867	0.2768
0.0274	0.1245	-0.0084	-0.0351	0.5497	-0.3977	0.4716	-0.1198	-0.1921	0.2419	0.0400	0.2526	-0.2066	0.2569	-0.1262
-0.0514	-0.4135	-0.3536	0.3842	0.3290	-0.0383	-0.2191	0.1593	0.1770	-0.2125	-0.3723	0.3512	-0.1216	-0.0296	0.1286
-0.1637	-0.2229	-0.3960	0.0381	-0.0507	0.2602	0.1995	-0.3910	-0.4284	0.0855	-0.2878	-0.3982	-0.1791	-0.1077	-0.1622
-0.0593	0.1981	-0.1484	-0.1802	-0.1174	0.3317	0.1457	-0.4009	0.2852	0.3212	-0.2205	0.4165	0.1531	0.0490	0.4057
0.3175	0.2793	-0.5255	-0.0288	-0.1842	-0.0569	-0.2807	-0.0425	-0.2962	-0.0798	0.3917	0.2489	-0.3203	-0.0435	0.1120
-0.1250	0.0256	0.4038	0.0662	0.2605	0.3063	-0.0452	-0.3501	-0.0866	-0.2777	0.2142	0.2922	-0.2229	-0.5009	-0.1168

Exponential of the skew symmetric matrix is an orthogonal matrix

More on orthogonal matrices.

Theorem:

If A is an orthogonal matrix, then $A^T = A^{-1}$. (Assume A is a $n \times n$ matrix)

Proof:

Since A is orthogonal it follows that $\det A = \pm 1 \Rightarrow \det A \neq 0$.
Hence A is invertible ie has rank n.
Let B be the inverse of matrix A i.e.

$$AB = BA = I$$

Since A is an orthogonal matrix, it follows that

$$AA^T = A^TA = I$$

Thus $B = A^{-1}$ and A^T are both inverses of the matrix A. However inverse of a matrix is unique Hence $A^{-1} = A^T$. ▨

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Fact:

Inverse of a matrix is unique.

Proof: Suppose B and C are two inverses of a matrix A ie.

$$\begin{aligned} AB &= BA = I \\ AC &= CA = I \end{aligned} \quad , \; B \neq C.$$

$$\Downarrow$$

$$A(B-C) = (B-C)A = I - I = 0$$

$$\because \; B \neq C \Rightarrow B - C \neq 0 \quad \star$$

However

$$A(B-C) = 0$$

$$\Rightarrow A^{-1}A(B-C) = A^{-1}0 = 0$$

$$\Rightarrow I(B-C) = B-C = 0 \quad \star\star$$

$\star$ & $\star\star$ are contradictory

Hence $B = C$. $\square$

Remark:

If P is an orthogonal matrix,

$P^T A P$ or $P A P^T$

are the associated similarity operation on A.

(As opposed to $P^{-1} A P$ or $P A P^{-1}$)

— x —

Let A be a symmetric matrix and

$v_1, v_2, \ldots, v_n$

be the eigenvectors of A, necessarily l. i. and mutually orthogonal.

"Recall that the associated eigenvalues are not necessarily distinct."

The matrix

$$P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$$

is orthogonal.

$$\boxed{P^T A P = \Lambda}$$

P is orthogonal.

A is symmetric

Λ is diagonal $= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Example:

1. Manufacture a 5×5 symmetric matrix A. Calculate the eigenvector matrix P and show that $P^T P = P P^T = I$.

2. Repeat 1 assuming A is symmetric and positive definite.

3. Consider the quadratic form

$$(x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5)\, A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0 \qquad (\star)$$

write down $(\star)$ as a quadratic equation.

4. Define new variables $(y_1, \ldots y_5)$ such that $(\star)$ is transformed as.

$$\alpha_1 y_1^2 + \alpha_2 y_2^2 + \alpha_3 y_3^2 + \alpha_4 y_4^2 + \alpha_5 y_5^2 = 0$$

This is an ellipsoid in $\mathbb{R}^5$.

Step 1: I am writing an arbitrary matrix 'A':

>> A=[2 4 6 5 9;3 8 5 1 9;4 3 5 7 1;5 3 1 8 9; 3 1 8 7 6]

A =

$$\begin{pmatrix} 2 & 4 & 6 & 5 & 9 \\ 3 & 8 & 5 & 1 & 9 \\ 4 & 3 & 5 & 7 & 1 \\ 5 & 3 & 1 & 8 & 9 \\ 3 & 1 & 8 & 7 & 6 \end{pmatrix}$$

Step 2: I am symmetrizing the matrix 'A':

>> A=[A+transpose(A)]/2
A =

$$\begin{pmatrix} 2.0000 & 3.5000 & 5.0000 & 5.0000 & 6.0000 \\ 3.5000 & 8.0000 & 4.0000 & 2.0000 & 5.0000 \\ 5.0000 & 4.0000 & 5.0000 & 4.0000 & 4.5000 \\ 5.0000 & 2.0000 & 4.0000 & 8.0000 & 8.0000 \\ 6.0000 & 5.0000 & 4.5000 & 8.0000 & 6.0000 \end{pmatrix}$$

Getting rid of the ugly decimals:

>> A=floor(A)

A =

$$\begin{pmatrix} 2 & 3 & 5 & 5 & 6 \\ 3 & 8 & 4 & 2 & 5 \\ 5 & 4 & 5 & 4 & 4 \\ 5 & 2 & 4 & 8 & 8 \\ 6 & 5 & 4 & 8 & 6 \end{pmatrix}$$

Step 3: Checking the eigenvalues to see if A is positive definite.

>> eig(A)
ans = -3.120, -1.1157, 2.1822, 6.3872, 24.6669
The conclusion is that 'A' is ***not positive definite.***

```
>> [v1 v2] = eig(A)
```

Step 4: Calculating the orthogonal matrix of eigenvectors 'v1'

v1 =

$$\begin{pmatrix} -0.7275 & 0.5023 & 0.2473 & 0.0649 & 0.3911 \\ -0.1431 & -0.2325 & -0.3451 & -0.8112 & 0.3853 \\ 0.3243 & -0.2169 & 0.8188 & -0.1580 & 0.3903 \\ -0.1686 & -0.5996 & -0.2316 & 0.5436 & 0.5127 \\ 0.5627 & 0.5358 & -0.3093 & 0.1316 & 0.5323 \end{pmatrix}$$

The diagonal matrix of eigenvalues 'v2'.

v2 =

$$\begin{pmatrix} -3.1207 & 0 & 0 & 0 & 0 \\ 0 & -1.1157 & 0 & 0 & 0 \\ 0 & 0 & 2.1822 & 0 & 0 \\ 0 & 0 & 0 & 6.3872 & 0 \\ 0 & 0 & 0 & 0 & 24.6669 \end{pmatrix}$$

Step 5 : Similarity transformation with orthogonal matrices.

```
>> transpose(v1)*A*v1
```

ans =

$$\begin{pmatrix} -3.1207 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -1.1157 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 2.1822 & -0.0000 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 6.3872 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & 24.6669 \end{pmatrix}$$

Eureka, we get a diagonal matrix.

Step 6: Checking to see that 'v1' is indeed orthogonal.

>> transpose(v1)*v1

ans =

$$\begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} = I$$

>> v1*transpose(v1)

ans =

$$\begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} = I$$

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Step1: We want to write a symmetric positive definite matrix.

Let us start with a diagonal matrix 'A'.

>> A=diag([1 2 3 4 5],0)

A =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Step2: Let us use the orthogonal matrix 'v1'.
>> v1 in page 8·35.

v1 =

$$\begin{pmatrix} -0.7275 & 0.5023 & 0.2473 & 0.0649 & 0.3911 \\ -0.1431 & -0.2325 & -0.3451 & -0.8112 & 0.3853 \\ 0.3243 & -0.2169 & 0.8188 & -0.1580 & 0.3903 \\ -0.1686 & -0.5996 & -0.2316 & 0.5436 & 0.5127 \\ 0.5627 & 0.5358 & -0.3093 & 0.1316 & 0.5323 \end{pmatrix}$$

Step 3: Construct a symmetric positive definite matrix 'B'.
>> B=v1*A*transpose(v1)

B =

$$\begin{pmatrix} 1.9991 & 0.1574 & 0.8759 & 0.4922 & 0.9745 \\ 0.1574 & 3.8599 & 0.4713 & -0.2335 & 0.5890 \\ 0.8759 & 0.4713 & 3.0723 & 0.2937 & 0.1460 \\ 0.4922 & -0.2335 & 0.2937 & 3.4050 & 1.1284 \\ 0.9745 & 0.5890 & 0.1460 & 1.1284 & 2.6637 \end{pmatrix}$$

Checking that 'B' has eigenvalues where it should be.

>> eig(B)

ans = 1.0000, 5.0000, 2.0000, 3.0000, 4.0000

Removing decimals from the matrix 'B'.
>> B=round(B)

B =

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 1 & 1 & 0 & 1 & 3 \end{pmatrix}$$

Eigenvalues of 'B' have changed but they are still all positive.
>> eig(B)

ans =

0.8256
2.1219
3.3697
3.7228
4.9600

Thus 'B' is still positive definite and symmetric.

>> [v1 v2]=eig(B)

The new set of eigenvectors of 'B'. The matrix 'v1' is an orthogonal matrix.

v1 =

$$\begin{pmatrix} -0.7292 & 0.3953 & 0.1902 & 0.4589 & 0.2554 \\ -0.1641 & -0.2654 & 0.4030 & -0.5618 & 0.6516 \\ 0.3353 & -0.4502 & 0.5146 & 0.6349 & 0.1303 \\ -0.2396 & -0.5676 & -0.6871 & 0.2154 & 0.3192 \\ 0.5210 & 0.4984 & -0.2540 & 0.1557 & 0.6256 \end{pmatrix}$$

v2 =

$$\begin{pmatrix} 0.8256 & 0 & 0 & 0 & 0 \\ 0 & 2.1219 & 0 & 0 & 0 \\ 0 & 0 & 3.3697 & 0 & 0 \\ 0 & 0 & 0 & 3.7228 & 0 \\ 0 & 0 & 0 & 0 & 4.9600 \end{pmatrix}$$

Let us construct a quadratic form given by

$$(x_1\ x_2\ x_3\ x_4\ x_5)\, B\, (x_1\ x_2\ x_3\ x_4\ x_5)^T = 0$$

$$\Rightarrow 2x_1^2 + 4x_2^2 + 3x_3^2 + 3x_4^2 + 3x_5^2$$
$$+ 0\,x_1x_2 + 2x_1x_3 + 0\,x_1x_4 + 2x_1x_5$$
$$+ 0\,x_2x_3 + 0\,x_2x_4 + 2x_2x_5$$
$$+ 0\,x_3x_4 + 0\,x_3x_5$$
$$+ 2x_4x_5$$
$$= 0$$

$$\Rightarrow 2x_1^2 + 4x_2^2 + 3x_3^2 + 3x_4^2 + 3x_5^2$$
$$+ 2x_1x_3 + 2x_1x_5 + 2x_2x_5 + 2x_4x_5 = 0$$

☆

$$V1^T\, B\, V1 = \quad V2$$

Let

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

We have

$$y^T (V2) y = \underbrace{\left(y^T (V1)^T\right)}_{x^T} B \underbrace{\left((V1)\, y\right)}_{x}$$

$$(V1)\, y = x \qquad \boxed{y = (V1)^T x}$$

We define new variables y using this relation.

The quadratic equation (A) reduces to

$$y^T (V2) y = 0$$

$$\Rightarrow .8256\, y_1^2 + 2.1219\, y_2^2 + 3.3697\, y_3^2 + 3.7228\, y_4^2 + 4.9600\, y_5^2 = 0.$$

Note that all the cross terms have been cancelled.